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The classification of Novikov algebras in low dimensions

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Abstract

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic-type and Hamiltonian operators in the formal variational calculus. For further our understanding and physical applications, we give a classification of Novikov algebras in dimensions two and three in this paper.

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1. Introduction

One remarkable feature of Hamiltonian operators is their connection with certain algebraic structures [1–8]. Gel'fand and Dikii introduced a formal variational calculus and found certain interesting Poisson structures when they studied Hamiltonian systems related to certain nonlinear partial differential equations, such as KdV equations [1, 2]. In [3], Gel'fand and Dorfman found more connections between Hamiltonian operators and certain algebraic structures. Dubrovin, Balanskii and Novikov studied similar Poisson structures from another point of view [4–6]. One of the algebraic structures appearing in [3, 6], which is called a 'Novikov algebra' by Osborn [9–13], was introduced in connection with the Poisson brackets of hydrodynamic type.

A Novikov algebra A is a vector space over a field $\boldsymbol{K}$ with a bilinear product $(x, y) \to xy$ satisfying

$$(x_1, x_2, x_3) = (x_2, x_1, x_3) \tag{1.1}$$

and

$$(x_1x_2)x_3 = (x_1x_3)x_2 \tag{1.2}$$

for $x_1, x_2, x_3 \in A$, where

$$(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3). \tag{1.3}$$

Novikov algebras are a special class of left-symmetric algebras which only satisfy equation (1.1). Left-symmetric algebras are non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [14–19].

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y] = xy - yx \tag{1.4}$$

defines a Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x, R_x denote the left and right multiplication, respectively, i.e. $L_x(y) = xy, R_x(y) = yx, \forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative.

Novikov asked whether there exist simple Novikov algebras (i.e. ones which contain no ideas except the zero ideal, itself and $AA \neq 0$). Zel'manov proved that a finite-dimensional simple Novikov algebra over an algebraically closed field with characteristic 0 is a field [20]. Osborn and Xu gave a complete classification of finite-dimensional simple Novikov algebras over an algebraically closed field with prime characteristic. They also found several classes of infinite-dimensional simple Novikov algebras [9–13].

Moreover, Zel'manov gave a fundamental structure theory of a finite-dimensional Novikov algebra over an algebraically closed field with characteristic 0 [20]: a Novikov algebra A is called right-nilpotent or transitive if every R_x is nilpotent. Then by equation (1.2), a finite-dimensional Novikov algebra contains the largest transitive ideal $N(A)$ and the quotient algebra $A/N(A)$ is a direct sum of fields. The transitivity corresponds to the completeness of the affine manifolds in geometry [14, 15].

Therefore, it is necessary to understand the structures and properties of transitive Novikov algebras in detail. This is still an open question, which is obviously quite difficult. Moreover, even if we can obtain some classifications of transitive Novikov algebras, we are still far from the complete classification of Novikov algebras. One of the most important reasons for this is that, unlike associative algebras, the extension by $N(A)$ is not non-essential in general. There can exist many non-isomorphic extensions. Recall that A is an extension of C by B if there exists an ideal R of A which is isomorphic to B and the quotient algebra A/R is isomorphic to C. If there exists a subalgebra H of A such that $R \cap H = \{0\}$ and $A = R + H$, then this extension is called a non-essential extension, otherwise it is called an essential extension. If, in addition, H is an ideal of A, then the extension is called a trivial extension.

In this paper, we give a classification of Novikov algebras over the complex field in dimensions two and three. As in the study of other algebras, while some special cases are understood, the full structure theory of Novikov algebras is yet to be developed. The study of the low-dimensional cases will serve as a guide for further development.

The paper is organized as follows. Section 2 gives the classification of two-dimensional Novikov algebras. Section 3 describes the classification of transitive Novikov algebras in dimension three. Section 4 describes the classification of non-transitive Novikov algebras in dimension three. In section 5 we give our discussion for the classifications used in the previous sections.

2. Two-dimensional Novikov algebras

Let $\{e_i\}$ be a basis of a Novikov algebra A. Set $A_{ij} = e_i e_j = \sum_{k=1}^n a_{ij}^k e_k$. Then the (form) characteristic matrix $\mathcal{A} = (A_{ij})$, i.e.

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^n a_{11}^k e_k & \cdots & \sum_{k=1}^n a_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^n a_{n1}^k e_k & \cdots & \sum_{k=1}^n a_{nn}^k e_k \end{pmatrix}. \tag{2.1}$$

Table 1. The classification and some basic properties of two-dimensional Novikov algebras.

Characteristic matrix	Associativity	Lie algebra $\mathcal{G}(A)$	$N(A)$	Ext. by $N(A)$
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	Associative	Abelian	$A = N(A)$	Transitive
(T2) $\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}$	Associative	Abelian	$A = N(A)$	Transitive
(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	Non-associative	$[e_1, e_2] = e_1$	$A = N(A)$	Transitive
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	Associative	Abelian	$N(A) = \{0\}$	$A \cong \boldsymbol{C} \oplus \boldsymbol{C}$
(N2) $\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}$	Associative	Abelian	(T0)	Trivial
(N3) $\begin{pmatrix} e_1 & e_2 \\ e_2 & 0 \end{pmatrix}$	Associative	Abelian	(T0)	Non-essential
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	Associative	Isomorphic to (T3)	(T0)	Non-essential
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	Non-associative	Isomorphic to (T3)	(T0)	Essential
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0, 1$	Non-associative	Isomorphic to (T3)	(T0)	Non-essential

There are two Novikov algebras in dimension one: the complex field $\boldsymbol{C} = \{\boldsymbol{C}e \mid ee = e\}$ and the one-dimensional trivial Novikov algebra (T0) $= \{\boldsymbol{C}e \mid ee = 0\}$. In [17, 19], the classification of two-dimensional left-symmetric algebras is given. Then by the condition $R_{e_1} R_{e_2} = R_{e_2} R_{e_1}$, we can obtain the classification of two-dimensional Novikov algebras given in table 1.

3. Three-dimensional transitive Novikov algebras

Let A be a three-dimensional transitive Novikov algebra. By Engel's theorem, there exists a basis $\{e_1, e_2, e_3\}$, such that $R_{e_1}, R_{e_2}, R_{e_3}$ can be put into strictly upper triangular matrices simultaneously, that is, we can assume

$$R_{e_1} = \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix} \qquad R_{e_2} = \begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix} \qquad R_{e_3} = \begin{pmatrix} 0 & a_3 & b_3 \\ 0 & 0 & c_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

By the commutativity of R_{e_i} and R_{e_j}, we have

$$a_1 c_2 = a_2 c_1 \qquad a_2 c_3 = a_3 c_2 \qquad a_1 c_3 = a_3 c_1. \tag{3.2}$$

Thus, we have

$$c_1 = c_2 = c_3 = 0 \tag{3.3}$$

or we can assume

$$a_1 = tc_1 \qquad a_2 = tc_2 \qquad a_3 = tc_3. \tag{3.4}$$

Through equation (1.1), we have the following equations:

$$
\begin{aligned}
(e_1, e_3, e_1) &= (e_3, e_1, e_1) &\Longrightarrow\quad & a_1c_1 = 0 &(3.5)\\
(e_1, e_3, e_2) &= (e_3, e_1, e_2) &\Longrightarrow\quad & a_2c_1 = 0 &(3.6)\\
(e_1, e_3, e_3) &= (e_3, e_1, e_3) &\Longrightarrow\quad & a_3c_1 = 0 &(3.7)\\
(e_2, e_3, e_1) &= (e_3, e_2, e_1) &\Longrightarrow\quad & a_1c_2 + a_2c_1 = 0 &(3.8)\\
(e_2, e_3, e_2) &= (e_3, e_2, e_2) &\Longrightarrow\quad & a_2b_1 - a_1b_2 - 2a_2c_2 = 0 &(3.9)\\
(e_2, e_3, e_3) &= (e_3, e_2, e_3) &\Longrightarrow\quad & a_3b_1 - a_1b_3 - a_2c_3 - a_3c_2 = 0. &(3.10)
\end{aligned}
$$

The relations for other items hold automatically. So the problem turns into one of how to determine the above parameters and the isomorphic classes of Novikov algebras that they define. From equations (3.3) and (3.4), we know $\{a_i, b_i, c_i\}$ must be in the following cases.

Case (I). $t = 0$. We will show that this implies $a_1 = a_2 = a_3 = 0$.

Case (II). $c_1 = c_2 = c_3 = 0$.

Case (III). One of a_i and one of c_j are not zero.

Next we discuss these three cases.

Case (I). This is the case where equation (3.4) holds and $t = 0$. Thus we have $a_1 = a_2 = a_3 = 0$. One verifies that equations (3.5)–(3.10) hold in this case. It is easy to see that the linear subspace V spanned by e_1 and e_2 is a trivial ideal. Hence V is stable under L_{e_3}. Then we can choose a new basis in V such that under this new basis L_{e_3} becomes

$$L_{e_3} = \begin{pmatrix} b_1' & 0 \\ 0 & c_2' \end{pmatrix} \qquad \text{or} \qquad L_{e_3} = \begin{pmatrix} b_1' & 1 \\ 0 & b_1' \end{pmatrix}. \tag{3.11}$$

The former shows that we can assume $c_1 = b_2 = 0$, and the latter case shows that we can assume $b_1 = c_2, c_1 = 0, b_2 = 1$. Next we discuss these two cases.

Case (I-1). The characteristic matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1e_1 & c_2e_2 & b_3e_1 + c_3e_2 \end{pmatrix}.$$

(i) $b_1 = c_2 = b_3 = c_3 = 0$. We have type (A1)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(ii) $b_1 = c_2 = 0$; $b_3 \neq 0$ or $c_3 \neq 0$. Let $e_1 \to b_3e_1 + c_3e_2$, we have type (A2)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix}.$$

(iii) $b_1 = 0, c_2 \neq 0$. Let $e_3 \to \frac{1}{c_2} e_3$, we can assume $c_2 = 1$.

(1) $b_3 = c_3 = 0$. We have type (A9)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}.$$

(2) $b_3 \neq 0, c_3 = 0$. Let $e_1 \to b_3 e_1$, we have type (A10)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}.$$

(3) $b_3 = 0, c_3 \neq 0$. Let $e_3 \to e_3 - c_3 e_2$, then this is case (I-1-iii-1).

(4) $b_3 \neq 0, c_3 \neq 0$. Let $e_1 \to b_3 e_1$; $e_3 \to e_3 - c_3 e_2$, then this is case (I-1-iii-2).

(iv) $b_1 \neq 0, c_2 = 0$. Let $e_1 \to e_2, e_2 \to e_1$, then this is case (I-1-iii);

(v) $b_1 \neq 0, c_2 \neq 0$. Let $e_3 \to \frac{1}{b_1} e_3 - \frac{b_3}{b_1^2} e_1 - \frac{c_3}{b_1 c_2} e_2$, then (non-zero products)

$$e_3 e_1 = e_1 \qquad e_3 e_2 = \frac{c_2}{b_1} e_2 \qquad e_3 e_3 = 0.$$

If $|\frac{c_2}{b_1}| > 1$, then after letting $e_1 \to e_2, e_2 \to e_1, e_3 \to \frac{b_1}{c_2} e_3$, we can assume $|l| \leqslant 1, l \neq 0$, where $l = \frac{c_2}{b_1}$. Thus we have type (A11)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & l e_2 & 0 \end{pmatrix}$$

with $|l| \leqslant 1, l \neq 0$.

Case (I-2). The characteristic matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1 e_1 & e_1 + b_1 e_2 & b_3 e_1 + c_3 e_2 \end{pmatrix}.$$

(i) $b_1 = 0$. Let $e_3 \to e_3 - b_3 e_2$, we can assume $b_3 = 0$.

(1) $c_3 = 0$. Let $e_2 \to e_2 + e_3, e_3 \to -e_3 + e_2$, we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & -e_1 \end{pmatrix}$$

which is type (A6) (see case (II-2-i-2)) with $l = -1$;

(2) $c_3 \neq 0$. Let $e_2 \to c_3 e_2$, we have type (A8)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & e_2 \end{pmatrix}.$$

(ii) $b_1 \neq 0$. Let $e_1 \to \frac{1}{b_1} e_1, e_3 \to \frac{1}{b_1} e_3 + \left(\frac{c_3}{b_1^3} - \frac{b_3}{b_1^2}\right) e_1 - \frac{c_3}{b_1^2} e_2$, we have type (A12)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & 0 \end{pmatrix}.$$

Case (II). This is the case where equation (3.3) holds. One can easily show that equations (3.5)–(3.8) hold. From equations (3.9) and (3.10), we have

$$a_2b_1 = a_1b_2 \qquad a_3b_1 = a_1b_3. \tag{3.12}$$

The characteristic matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ a_1e_1 & a_2e_1 & a_3e_1 \\ b_1e_1 & b_2e_1 & b_3e_1 \end{pmatrix}.$$

We can assume one of a_i is not zero.

Case (II-1). $b_1 = b_2 = b_3 = 0$. Let $e_2 \to e_3, e_3 \to e_2$, Then we can see this is in case (I), i.e. $a_1 = a_2 = a_3 = 0$.

Case (II-2). $b_1 = 0$, and one of b_2 and b_3 is not zero: by equation (3.12), we have $a_1b_2 = a_1b_3 = 0$. Hence $a_1 = 0$, otherwise $b_2 = b_3 = 0$. Note $a_2 \neq 0$ or $a_3 \neq 0$ by our assumption.

(i) $a_2 \neq 0$. Let $e_1 \to a_2e_1$, then the characteristic matrix can be written as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & a_3e_1 \\ 0 & b_2e_1 & b_3e_1 \end{pmatrix}.$$

(1) $a_3 = b_2$. Let $e_3 \to e_3 - b_2e_2$, we can assume $a_3 = b_2 = 0$. Thus, by our assumption, $b_3 \neq 0$. Let $e_3 \to \sqrt{b_3}e_3$, we have type (A3)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}.$$

(2) $a_3 \neq b_2$. Let $e_3 \to \frac{2}{a_3-b_2}e_3 - \frac{a_3+b_2}{a_3-b_2}e_2$, we have type (A6)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & le_1 \end{pmatrix}$$

where

$$l = \frac{4b_3 - (a_3 + b_2)^2}{(a_3 - b_2)^2}.$$

(ii) $a_2 = 0$. We can assume $b_3 = 0$, otherwise, after letting $e_2 \to e_3, e_3 \to e_2$, this case is changed into case (II-2-i).

(1) $a_3 = -b_2 \neq 0$. Let $e_2 \to e_2 + e_3, e_3 \to \frac{1}{2a_3}(e_3 - e_2)$, we have type (A5)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}.$$

(2) $a_3 + b_2 \neq 0$. Let $e_2 \to e_2 + e_3$, then this is in case (II-2-i).

Case (II-3) $b_1 \neq 0$.

(i) $b_2 = 0$. Then by equation (3.12), $a_2 = 0$. Moreover, $a_1 \neq 0$, otherwise $a_1 = a_2 = a_3 = 0$.

(1) $a_3 = 0$. By equation (3.12), $b_3 = 0$. Let $e_1 \to e_2 - \frac{a_1}{b_1}e_3$, $e_2 \to e_1$, $e_3 \to \frac{1}{b_1}e_3$, this is case (I-1-iii-1), i.e. type A(9).

(2) $a_3 \neq 0$. Let $e_2 \to e_2 - \frac{a_1}{b_1}e_3$. Since $\frac{b_3}{a_3} = \frac{b_1}{a_1} \neq 0$, this is in case (I).

(ii) $b_2 \neq 0$. If $b_3 = 0$, then by equation (3.12), we have $a_3 = 0$. Moreover, $a_1 \neq 0, a_2 \neq 0$, otherwise $a_1 = a_2 = a_3 = 0$. Let $e_3 \to e_2$, $e_2 \to e_3$, this is in case (I) since $\frac{b_1}{a_1} = \frac{b_2}{a_2} \neq 0$. If $b_3 \neq 0$, then $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$, otherwise $a_1 = a_2 = a_3 = 0$. Let $e_2 \to e_2 - \frac{a_1}{b_1}e_3$, this is still in case (I) since $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} \neq 0$.

Case (III). This is the case where equation (3.4) holds and $t \neq 0$, and one of c_i is not zero. And one of a_i is not zero, too. Then we have $c_1 = 0$ by equations (3.5)–(3.7). Hence $a_1 = 0$ and $b_1 = 2c_2$ by equation (3.4) and equations (3.8)–(3.10). We assume c_i cannot be zero at all.

Case (III-1) $c_2 = b_1 = 0$. In this case, we have $a_2 = 0$ by equation (3.4) and $c_3 \neq 0$ by our assumption. Let $e_1 \to tc_3e_1$, the characteristic matrix can be assumed to be

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & b_2e_1 & b_3e_1 + c_3e_2 \end{pmatrix}.$$

(i) $b_2 \neq -1$. Let $e_3 \to e_3 - \frac{b_3}{b_2+1}e_2$, $e_2 \to c_3e_2$, then the characteristic matrix can be written as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & b_2e_1 & e_2 \end{pmatrix}.$$

We divide them into two classes: associative case only and only if $b_2 = 1$, which is type (A4)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & e_2 \end{pmatrix}$$

and non-associative case only and only if $b_2 \neq 1$, which is type (A7)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & le_1 & e_2 \end{pmatrix}$$

with $l = b_2 \neq 1, -1$.

(ii) $b_2 = -1$. Let $e_1 \to c_3e_1$, $e_2 \to c_3e_2 + b_3e_1$, we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1 & e_2 \end{pmatrix}$$

which is type (A7) with $l = -1$.

Table 2. The classification and some basic properties of three-dimensional transitive Novikov algebras.

Characteristic matrix	Associativity	Lie algebra $\mathcal{G}(A)$	Remark
(A1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Associative	Abelian	(T0) $\oplus$ (T1)
(A2) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix}$	Associative	Abelian	(T0) $\oplus$ (T2)
(A3) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}$	Associative	Abelian	
(A4) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & e_2 \end{pmatrix}$	Associative	Abelian	
(A5) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}$	Associative	$[e_2, e_1] = 0$ $[e_3, e_1] = 0$ $[e_3, e_2] = e_1$	
(A6) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & le_1 \end{pmatrix}$	Associative	Isomorphic to (A5)	
(A7) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & le_1 & e_2 \end{pmatrix}$ $l \neq 1$	Non-associative	Isomorphic to (A5)	
(A8) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & e_2 \end{pmatrix}$	Non-associative	Isomorphic to (A5)	
(A9) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}$	Non-associative	$[e_2, e_1] = 0$ $[e_3, e_1] = 0$ $[e_3, e_2] = e_2$	(T0) $\oplus$ (T3)
(A10) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}$	Non-associative	Isomorphic to (A9)	
(A11) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 0 \end{pmatrix}$ $\lvert l\rvert \leqslant 1, l \neq 0$	Non-associative	$[e_2, e_1] = 0$ $[e_3, e_1] = e_1$ $[e_3, e_2] = le_2$ $\lvert l\rvert \leqslant 1, l \neq 0$	
(A12) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & 0 \end{pmatrix}$	Non-associative	$[e_2, e_1] = 0$ $[e_3, e_1] = e_1$ $[e_3, e_2] = e_1 + e_2$	
(A13) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 & 0 \end{pmatrix}$	Non-associative	$[e_2, e_1] = 0$ $[e_3, e_1] = e_1$ $[e_3, e_2] = \frac{1}{2}e_2$	$\mathcal{G}(A)$ is (A11) with $l = \frac{1}{2}$

Case (III-2) $b_1 = 2c_2 \neq 0$. Let $e_1 \to tc_2e_1, e_3 \to \frac{1}{2c_2}e_3 - \frac{b_3}{4c_2^2}e_1$, then the characteristic matrix can be written as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & c_3 \,!/\, e_1 \\ e_1 & b_2 \,!/\, e_1 + \frac{1}{2}e_2 & \frac{c_3 \,!/}{2}e_2 \end{pmatrix}$$

where $b_2 \,!/ = \frac{b_2}{2c_2}, c_3 \,!/ = \frac{c_3}{2c_2^2}$.

(i) c_3 !/ $= 0$.

(1) b_2 !/ $= 0$. We have type (A13),

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 & 0 \end{pmatrix}.$$

(2) b_2 !/ $\neq 0$. Let $e_2 \to b_2$!/ $e_2 - 2b_2$!/ $^2 e_1$, $e_1 \to b_2$!/ $^2 e_1$, then it is isomorphic to type (A13).

(ii) c_3 !/ $\neq 0$. Let $e_3 \to e_3 + b_2$!/ c_3 !/ $e_1 - c_3$!/ e_2, then

$$e_2e_3 = e_3e_3 = 0 \qquad e_2e_3 = (b_2 \text{ !/ } - c_3 \text{ !/ })e_1 + \tfrac{1}{2}e_2 \qquad e_1e_3 = 0 \qquad e_3e_1 = e_1.$$

This is in case (III-2-i).

We summarize the above results in table 2. In the second and the third columns, we indicate the associativity of the algebra and the structure of the Lie algebra $\mathcal{G}(A)$. One can show the algebras in the table that are mutually non-isomorphic.

4. Three-dimensional non-transitive Novikov algebras

Using the introduction, we discuss the three-dimensional non-transitive Novikov algebras in the following cases: the cases $N(A) = \{0\}$, $N(A) =$ (T0), $N(A) =$ (T1), $N(A) =$ (T2) and $N(A) =$ (T3), respectively.

Case (1) $N(A) = \{0\}$. In this case, there is only one Novikov structure, that is, the direct sum of the fields

$$A = \boldsymbol{C} \oplus \boldsymbol{C} \oplus \boldsymbol{C}. \tag{4.1}$$

Case (2) $N(A) =$ (T0). We can choose a basis e_1, e_2, e_3 such that

$$e_1e_1 = 0 \qquad e_1e_i = a_{1i}e_1 \qquad e_ie_1 = a_{i1}e_1 \qquad e_ie_j = \delta_{ij}e_i + a_{ij}e_1 \qquad i = 2, 3 \tag{4.2}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Then we have

$$R_{e_1}R_{e_2} = R_{e_2}R_{e_1} \quad\Longrightarrow\quad (a_{12} - 1)a_{21} = 0 \quad a_{31}a_{12} = 0 \tag{4.3}$$

$$R_{e_1}R_{e_3} = R_{e_3}R_{e_1} \quad\Longrightarrow\quad (a_{13} - 1)a_{31} = 0 \quad a_{21}a_{13} = 0 \tag{4.4}$$

$$R_{e_2}R_{e_3} = R_{e_3}R_{e_2} \quad\Longrightarrow\quad (a_{12} - 1)a_{23} = a_{13}a_{22} \quad (a_{13} - 1)a_{32} = a_{12}a_{33} \tag{4.5}$$

$$(e_1, e_2, e_2) = (e_2, e_1, e_2) \quad\Longrightarrow\quad a_{12}^2 = a_{12} \tag{4.6}$$

$$(e_1, e_2, e_3) = (e_2, e_1, e_3) \quad\Longrightarrow\quad a_{12}a_{13} = 0 \tag{4.7}$$

$$(e_1, e_3, e_3) = (e_3, e_1, e_3) \quad\Longrightarrow\quad a_{13}^2 = a_{13} \tag{4.8}$$

$$(e_2, e_3, e_2) = (e_3, e_2, e_2) \quad\Longrightarrow\quad a_{23}a_{23} - a_{21}a_{32} = a_{32}a_{12} - a_{32} - a_{22}a_{31} \tag{4.9}$$

$$(e_2, e_3, e_3) = (e_3, e_2, e_3) \quad\Longrightarrow\quad a_{13}a_{32} - a_{23}a_{31} = a_{23}a_{13} - a_{23} - a_{21}a_{33}. \tag{4.10}$$

The relations for other items hold automatically. Using the same method as in section 3, we can give the classification in table 3.

Table 3. The classification and some basic properties of three-dimensional Novikov algebras with $N(A) = $ (T0).

Characteristic matrix	Associativity	Lie algebra $\mathcal{G}(A)$	Ext. by $N(A)$	Remark
(B1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	Associative	Abelian	Trivial	(N2) $\oplus \boldsymbol{C}$
(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	Associative	Abelian	Non-essential	(N3) $\oplus \boldsymbol{C}$
(B3) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	Associative	Isomorphic to (A9)	Non-essential	(N4) $\oplus \boldsymbol{C}$
(B4) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_1 + e_3 \end{pmatrix}$	Non-associative	Isomorphic to (A9)	Essential	(N5) $\oplus \boldsymbol{C}$
(B5) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 0, 1$	Non-associative	Isomorphic to (A9)	Non-essential	(N6) $\oplus \boldsymbol{C}$

Case (3) $N(A) = $ (T1). We can choose a basis e_1, e_2, e_3 such that

$$e_1e_1 = e_1e_2 = e_2e_1 = e_2e_2 = 0 \qquad e_3e_3 = e_3 + a_{33}e_1 + b_{33}e_2$$
$$e_ie_3 = a_{i3}e_1 + b_{i3}e_2 \qquad e_3e_i = a_{3i}e_1 + b_{3i}e_2 \qquad i = 1, 2. \tag{4.11}$$

Then we have

$$R_{e_1}R_{e_3} = R_{e_3}R_{e_1} \quad \Longrightarrow \quad a_{13}a_{32} + a_{23}b_{32} = a_{32} \quad a_{32}b_{13} + b_{23}b_{32} = b_{32} \tag{4.12}$$
$$R_{e_2}R_{e_3} = R_{e_3}R_{e_2} \quad \Longrightarrow \quad a_{13}a_{31} + a_{23}b_{31} = a_{31} \quad a_{31}b_{13} + b_{23}b_{31} = b_{31}. \tag{4.13}$$

By the relation $(e_1, e_3, e_3) = (e_3, e_1, e_3)$, we have

$$a_{13}^2 + a_{23}b_{13} - a_{13} = a_{23}b_{31} - a_{32}b_{13}$$
$$b_{13}(a_{13} + b_{23} - a_{31} + b_{32} - 1) = b_{31}(b_{23} - a_{13}). \tag{4.14}$$

By the relation $(e_2, e_3, e_3) = (e_3, e_2, e_3)$, we have

$$b_{23}^2 + a_{23}b_{13} - b_{23} = a_{32}b_{13} - a_{23}b_{31}$$
$$a_{23}(a_{13} + b_{23} + a_{31} - b_{32} - 1) = a_{32}(-b_{23} + a_{13}). \tag{4.15}$$

The relations for other items hold automatically. However, the above equations are still complicated. Note that $N(A)$ is an ideal of A, thus $N(A)$ is a stable subspace of the linear transformations L_{e_3} and R_{e_3}. Because the field is complex by our supposition, there exists $e_1', e_2' \in N(A)$, such that

$$e_3e_1' = \alpha e_1' \qquad e_2'e_3 = \beta e_2'. \tag{4.16}$$

Hence by a linear transformation $e_1 \to e_1', e_2 \to e_2'$, we can suppose that $b_{31} = a_{23} = 0$. So equations (4.12)–(4.15) become

$$(a_{13} - 1)a_{32} = (a_{13} - 1)a_{31} = a_{32}(a_{13} - b_{23}) = b_{13}a_{31} = 0 \tag{4.17}$$
$$-a_{13}(a_{13} - 1) = b_{23}(b_{23} - 1) = -b_{32}(b_{23} - 1) = b_{13}a_{32} \tag{4.18}$$
$$b_{13}(a_{13} + b_{23} - a_{31} + b_{32} - 1) = 0. \tag{4.19}$$

Using the same method as in section 3, we can give the classification in table 4.

Table 4. The classification and some basic properties of three-dimensional Novikov algebras with $N(A) = $ (T1).

Characteristic matrix	Associativity	Lie algebra $\mathcal{G}(A)$	Ext. by $N(A)$	Remark
(C1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	Associative	Abelian	Trivial	(N2) $\oplus$(T0)
(C2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	Associative	Abelian	Non-essential	(N3) $\oplus$ (T0)
(C3) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	Associative	Isomorphic to (A9)	Non-essential	(N4) $\oplus$ (T0)
(C4) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 + e_3 \end{pmatrix}$	Non-associative	Isomorphic to (A9)	Essential	(N5) $\oplus$ (T0)
(C5) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 0, 1$	Non-associative	Isomorphic to (A9)	Non-essential	(N6) $\oplus$ (T0)
(C6) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 \end{pmatrix}$	Associative	Isomorphic to (A9)	Non-essential	
(C7) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 + e_2 \end{pmatrix}$	Non-associative	Isomorphic to (A9)	Essential	
(C8) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & 0 & e_3 \end{pmatrix}$	Associative	$[e_2, e_1] = 0$ $[e_3, e_1] = e_1$ $[e_3, e_2] = e_2$	Non-essential	$\mathcal{G}(A)$ is (A11) with $l = 1$
(C9) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 1, 0$	Non-associative	$[e_2, e_1] = 0$ $[e_3, e_1] = le_1$ $[e_3, e_2] = e_2$ $\lvert l\rvert \leqslant 1, l \neq 0, 1$	Non-essential	$\mathcal{G}(A)$ is (A11) with $l \neq 1$
(C10) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & 0 & e_3 + e_2 \end{pmatrix}$ $l \neq 1$	Non-associative	Isomorphic to (A11)	Essential	
(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	Associative	Abelian	Non-essential	
(C12) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & le_2 & e_3 \end{pmatrix}$ $l \neq 0, 1$	Non-associative	Isomorphic to (A9)	Non-essential	
(C13) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & ke_2 & e_3 \end{pmatrix}$ $l, k \neq 1, 0$	Non-associative	Isomorphic to (A11)	Non-essential	
(C14) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_1 + e_2 & e_3 \end{pmatrix}$	Non-associative	Isomorphic to A(5)	Non-essential	
(C15) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & e_1 + le_2 & e_3 \end{pmatrix}$ $l \neq 1, 0$	Non-associative	Isomorphic to (A12)	Non-essential	
(C16) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3 \end{pmatrix}$	Non-associative	Isomorphic to A(12)	Non-essential	

Table 4. Continued.

Characteristic matrix	Associativity	Lie algebra $\mathcal{G}(A)$	Ext. by $N(A)$	Remark
(C17) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3+e_2 \end{pmatrix}$	Non-associative	Isomorphic to A(12)	Essential	
(C18) $\begin{pmatrix} 0 & 0 & e_1+e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3 \end{pmatrix}$	Non-associative	Isomorphic to A(13)	Non-essential	
(C19) $\begin{pmatrix} 0 & 0 & e_1+e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3+e_1 \end{pmatrix}$	Non-associative	Isomorphic to A(13)	Essential	

Case (4) $N(A) = (\mathrm{T2})$. We can choose a basis e_1, e_2, e_3 such that

$$\begin{aligned} &e_1e_2 = e_2e_1 = e_2e_2 = 0 \qquad e_1e_1 = e_2 \qquad e_3e_3 = e_3 + a_{33}e_1 + b_{33}e_2 \\ &e_ie_3 = a_{i3}e_1 + b_{i3}e_2 \qquad e_3e_i = a_{3i}e_1 + b_{3i}e_2 \qquad i = 1, 2. \end{aligned} \tag{4.20}$$

Then we have

$$R_{e_1}R_{e_2} = R_{e_2}R_{e_1} \quad \Longrightarrow \quad a_{32} = 0 \tag{4.21}$$

$$R_{e_2}R_{e_3} = R_{e_3}R_{e_2} \quad \Longrightarrow \quad a_{13}a_{32} + a_{23}b_{32} = a_{32} \quad a_{32}b_{13} + b_{23}b_{32} = b_{32} \tag{4.22}$$

$$R_{e_1}R_{e_3} = R_{e_3}R_{e_1} \quad \Longrightarrow \quad \begin{cases} a_{23} = 0 \quad a_{13} = b_{23} \\ a_{13}a_{31} + a_{23}b_{31} = a_{31} \\ a_{31}b_{13} + b_{23}b_{31} = b_{31} + a_{33} \end{cases} \tag{4.23}$$

$$(e_1, e_3, e_1) = (e_3, e_1, e_1) \quad \Longrightarrow \quad b_{23} + b_{32} = 2a_{31} \tag{4.24}$$

$$(e_1, e_3, e_3) = (e_3, e_1, e_3) \quad \Longrightarrow \quad b_{23}^2 - b_{23} = 0 \quad b_{13}(2b_{23} - a_{31} + b_{32} - 1) = a_{33}. \tag{4.25}$$

The relations for other items hold automatically. Using the same method as in section 3, we can give the classification in table 5.

Case (5) $N(A) = (\mathrm{T3})$. We can choose a basis e_1, e_2, e_3 such that

$$\begin{aligned} &e_1e_1 = e_1e_2 = e_2e_2 = 0 \qquad e_2e_1 = -e_1 \qquad e_3e_3 = e_3 + a_{33}e_1 + b_{33}e_2 \\ &e_ie_3 = a_{i3}e_1 + b_{i3}e_2 \qquad e_3e_i = a_{3i}e_1 + b_{3i}e_2 \qquad i = 1, 2. \end{aligned} \tag{4.26}$$

Then we have

$$R_{e_1}R_{e_2} = R_{e_2}R_{e_1} \quad \Longrightarrow \quad b_{32} = 0 \tag{4.27}$$

$$R_{e_2}R_{e_3} = R_{e_3}R_{e_2} \quad \Longrightarrow \quad a_{13}a_{32} + a_{23}b_{32} = a_{32} \quad a_{32}b_{13} + b_{23}b_{32} = b_{32} \tag{4.28}$$

$$R_{e_1}R_{e_3} = R_{e_3}R_{e_1} \quad \Longrightarrow \quad \begin{cases} b_{13} = 0 \quad a_{13} = b_{23} \\ a_{13}a_{31} + a_{23}b_{31} = a_{31} - b_{33} \\ a_{31}b_{13} + b_{23}b_{31} = b_{31} \end{cases} \tag{4.29}$$

$$(e_1, e_3, e_1) = (e_3, e_1, e_1) \quad \Longrightarrow \quad b_{31} = 0 \tag{4.30}$$

$$(e_1, e_3, e_3) = (e_3, e_1, e_3) \quad \Longrightarrow \quad b_{23}^2 - b_{23} = b_{31}a_{23} \tag{4.31}$$

$$(e_2, e_3, e_1) = (e_3, e_2, e_1) \quad \Longrightarrow \quad b_{23} = 0 \tag{4.32}$$

$$(e_2, e_3, e_3) = (e_3, e_2, e_3) \quad \Longrightarrow \quad a_{23}(2b_{23} + a_{31} - 1) = -a_{33} - b_{23}a_{32}. \tag{4.33}$$

Table 5. The classification and some basic properties of three-dimensional Novikov algebras with $N(A) = $ (T2).

Characteristic matrix	Associativity	Lie algebra $\mathcal{G}(A)$	Ext. by $N(A)$	Remark
(D1) $\begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	Associative	Abelian	Trivial	(T2) $\oplus$ $\boldsymbol{C}$
(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	Associative	Abelian	Non-essential	
(D3) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1+e_2 & e_2 & e_3 \end{pmatrix}$	Non-associative	Isomorphic to (A5)	Non-essential	
(D4) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{1}{2}e_1 & 0 & e_3 \end{pmatrix}$	Non-associative	Isomorphic to (A13)	Non-essential	
(D5) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{1}{2}e_1 & 0 & e_3+e_2 \end{pmatrix}$	Non-associative	Isomorphic to (A13)	Essential	
(D6) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & (2l-1)e_2 & e_3 \end{pmatrix}$ $l \neq \frac{1}{2}, 1$	Non-associative	Isomorphic to (A13)	Non-essential	

Table 6. The classification and some basic properties of three-dimensional Novikov algebras with $N(A) = $ (T3).

Characteristic matrix	Associativity	Lie algebra $\mathcal{G}(A)$	Ext. by $N(A)$	Remark
(E1) $\begin{pmatrix} 0 & 0 & 0 \\ -e_1 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	Non-associative	Isomorphic to A(6)	Trivial	(T3) $\oplus$ $\boldsymbol{C}$

The relations for other items hold automatically. Hence, equations (4.27)–(4.33) become

$$a_{13} = b_{13} = b_{31} = b_{23} = a_{32} = b_{32} = 0 \qquad a_{33} = a_{23}(1 - a_{31}) \qquad b_{33} = a_{31}. \tag{4.34}$$

Let

$$e_1' = e_1 \qquad e_2' = e_2 \qquad e_3' = e_3 + a_{31}e_2 + a_{23}e_1. \tag{4.35}$$

Then A is isomorphic to table 6.

5. Conclusion and discussion

From the classification of Novikov algebras in dimensions two and three given in the previous sections, we can obtain the following:

1. all three-dimensional left-symmetric transitive algebras whose sub-adjacent Lie algebras are nilpotent are Novikov algebras [14];
2. there exists a one-dimensional trivial ideal in any three-dimensional non-semisimple Novikov algebra;
3. unlike some other special left-symmetric algebras [17–19], according to the classification of three-dimensional Lie algebras, there exist (transitive) Novikov algebra structures in any three-dimensional non-semisimple Lie algebra.

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